AN EXTENSION OF THE INTUITIONISTIC PROPOSITIONAL CALCULUS

BY

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Recently D. McCULLOUGH has shown [2] that a certain class of propositional connectives, called *regular* and defined in terms of Kripke's semantical models, were all definable in the usual intuitionistic propositional calculus. This class of connectives has a certain naturalness to it and might serve as the starting point of an attempt to (classically) answer the question: What is an intuitionistic propositional connective? Our intention here is to introduce an extension of LJ_p, the propositional portion of Gentzen's intuitionistic sequentzen system LJ [1]. This extension incorporates additional propositional connectives together with rules characterizing them, and is such that these new connectives cannot be defined in terms of the original connectives $\neg$, $\supset$, $\vee$, and $\wedge$ (nor for that matter in terms of each other). As a consequence, no interpretation of this calculus in which these connectives are regular can be complete with respect to Kripke's semantics. The principal tool in this proof is the extension of Gentzen's Hauptsatz to the present system. This is then used to show that the connectives form an independent set.

As indicated above, the system LJ_p is the propositional portion of Gentzen's LJ. It has an infinite list of propositional variables $p, q, r, s, p_0, \ldots$, together with the connectives $\vee$, $\wedge$, $\supset$, and $\neg$, and the auxilliary symbol $\Rightarrow$. Formulas are formed as usual, while sequents are expressions of the form

$$A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m,$$

where the A_i and B_j are formulas, $n \geqslant 0$, and $0 \leqslant m \leqslant 1$. Proofs are given in tree form, using axioms of the form $A \Rightarrow A$. The rules of inference are the structural rules in III. 1.21 of [1] and the rules for introduction of propositional connectives in III. 1.22 of [1]. The Hauptsatz for LJ_p is the statement that the structural rule Cut is superfluous.

We will begin by considering the system $LJ_{\not\subset}$ obtained from LJ_p by adding the binary connective $\not\subset$ (*converse non-implication*) together with the following introduction rules:

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$\not\subset$-IS: $$\frac{A,\Gamma\Rightarrow \qquad \Gamma\Rightarrow B}{\Gamma\Rightarrow A\not\subset B}$$

$\not\subset$-IA: $$\frac{B,\Gamma\Rightarrow A}{A\not\subset B,\Gamma\Rightarrow}.$$

Sequents, axioms, and proofs are just as for LJ_p.

As in the case of LJ_p, we prove the Hauptsatz not by showing directly that the Cut rule is redundant, but that an equivalent rule Mix (cf. III. 3.1 of [1]) can be eliminated. The structure of the proof is the same as that for LJ_p, consisting of inductions on the *rank* ϱ and *degree* γ of the given proof. (Cf. [1] for the full definitions of these terms. The degree of $A\not\subset B$ is the same as the degree of $A\supset B$). Most of the argument is unaltered; we need only consider the cases in which the last inferences before the Mix to be eliminated are $\not\subset$-IS or $\not\subset$-IA.

Case 1: $\varrho=2$ and the inference is of the form:

$$\cfrac{\cfrac{A,\Gamma\Rightarrow \qquad \Gamma\Rightarrow B}{\Gamma\Rightarrow A\not\subset B}\ \not\subset\text{-}IS \qquad \cfrac{B,\Gamma\Rightarrow A}{A\not\subset B,\Gamma\Rightarrow}}{\Gamma\Rightarrow .}\ Mix$$

This is replaced by:

$$\cfrac{\cfrac{\cfrac{\Gamma\Rightarrow B \qquad B,\Gamma\Rightarrow A}{\Gamma^*\Rightarrow A}\ Mix \qquad A,\Gamma\Rightarrow}{\Gamma^{**}\Rightarrow}\ Mix}{\Gamma\Rightarrow}\ \text{possibly several thinnings and interchanges.}$$

Both mixes here are of lower degree than the original.

Case 2: $\varrho>2$.

Subcase 2.1: The right rank is >1 and the inference is:

$$\cfrac{\Pi\Rightarrow A\not\subset B \qquad \cfrac{B,\Gamma\Rightarrow A}{A\not\subset B,\Gamma\Rightarrow}\ \not\subset\text{-}IA}{\Pi,\Gamma^*\Rightarrow .}\ Mix$$

Since the right rank is >1, we can replace this by:

$$\cfrac{\cfrac{\Pi\Rightarrow A\not\subset B \qquad \cfrac{\cfrac{\cfrac{\Pi\Rightarrow A\not\subset B \qquad B,\Gamma\Rightarrow A}{\Pi,B,\Gamma^*\Rightarrow A}\ Mix\ (\text{on } A\not\subset B)}{B,\Pi,\Gamma^*\Rightarrow A}\ \text{possibly several interchanges}}{A\not\subset B,\Pi,\Gamma^*\Rightarrow}\ \not\subset\text{-}IA}{\Pi,\Pi^*,\Gamma^*\Rightarrow}\ Mix}{\Pi,\Gamma^*\Rightarrow}\ \text{possibly several interchanges and contractions.}$$

Both mixes have lower right rank than the original.

Subcase 2.2: The right rank $=1$, the left rank is >1, and the inference is:

$$\cfrac{\cfrac{B, \Gamma \Rightarrow \Delta, A}{A \not\subset B, \Gamma \Rightarrow \Delta}\ \not\subset\text{-}IA \qquad \Pi \Rightarrow \Sigma}{A \not\subset B, \Gamma, \Pi^* \Rightarrow \Delta^*, \Sigma.}\ Mix$$

This case cannot occur since Δ must be empty and so there could be no mix-formula.

Subcase 2.3: The right rank $=1$, the left rank is >1, and the inference is:

$$\cfrac{\cfrac{A, \Gamma \Rightarrow \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \not\subset B}\ \not\subset\text{-}IS \qquad \Pi \Rightarrow \Sigma}{\Gamma, \Pi^* \Rightarrow \Sigma.}\ Mix$$

This case also cannot occur since the left rank of this figure is clearly 1.

Thus we have:

Theorem 1. The Cut rule is superfluous in $LJ_{\not\subset}$.

To show that the connectives of $LJ_{\not\subset}$ are independent in $LJ_{\not\subset}$, we first prove a series of lemmas.

Lemma A. Any provable sequent of $LJ_{\not\subset}$ has a cut-free proof consisting only of *reduced sequents*, where a sequent is *reduced* if no formula occurs in the antecedent more than three times.

Proof. This is an extension of Lemma IV. 1.21 of [1] and is proved in the same manner.

Let T and F be the usual (classical) truth-values, and define $p \not\subset q$ to take value T when $p=F$ and $q=T$, and value F otherwise. We will call a sequent $A_1, \ldots, A_n \Rightarrow B$, $\Rightarrow B$, or $A_1, \ldots, A_n \Rightarrow$ a *tautology* provided the corresponding formula $A_1 \wedge \ldots \wedge A_n \supset B$, B, or $A_1 \wedge \ldots \wedge A_n \supset (p \wedge \neg p)$, respectively, is a tautology in the usual sense. Then it is easy to verify the following.

Lemma B. Every provable sequent of $LJ_{\not\subset}$ is a tautology.

Corollary. The following sequents are not provable in $LJ_{\not\subset}$:

$$p \Rightarrow q, \quad p \Rightarrow, \quad \Rightarrow p, \quad p \not\subset q \Rightarrow, \quad \Rightarrow p \not\subset q,$$
$$p \Rightarrow p \not\subset q, \quad q, p \not\subset q \Rightarrow, \quad q \Rightarrow p \not\subset q, \quad p \not\subset q \Rightarrow p.$$

Lemma C. $p \not\subset q \Rightarrow r$ is not provable even if r is q.

Proof. By Lemma A, if $p \not\subset q \Rightarrow r$ were provable, it would have a cut-free proof consisting of reduced sequents. By Lemma B, the last inference could not be a thinning, and since sequents in $LJ_{\not\subset}$ can have at most one formula in the consequent, the last inference could not be $\not\subset$-IA. This leaves contraction as the only possibility:

$$\frac{p \not\subset q,\ \ p \not\subset q \Rightarrow r}{p \not\subset q \Rightarrow r}.$$

But a similar argument shows that the only viable possibility for the last inference to the upper sequent is again a contraction:

$$\cfrac{\cfrac{p \not\subset q, \quad p \not\subset q, \quad p \not\subset q \Rightarrow r}{p \not\subset q, \quad p \not\subset q, \Rightarrow r}}{p \not\subset q \Rightarrow r.}$$

But by Lemma A, this iteration can be carried no further. Then the only possible inference to the uppermost sequent would be a thinning on the left which is circular. Hence $p \not\subset q \Rightarrow r$ is not provable in $LJ_{\not\subset}$.

Lemma D. Let p and q be distinct variables and let $A_1, \ldots, A_n$ be distinct formulae containing no occurrences of the symbol $\not\subset$. Then not all of the following sequents can be provable:

$$A_1, \ldots, A_n \Rightarrow p \not\subset q$$
$$p \not\subset q \Rightarrow A_i, \qquad i = 1, \ldots, n.$$

Proof. We proceed by induction on the total number, K, of logical symbols occurring in the A_i. If $K=0$, the result follows immediately from Lemma C. So assume that $K>0$ and that some A_i, say A_1, is of the form $B \wedge C$, and suppose the Lemma false. Then in particular $p \not\subset q \Rightarrow B \wedge C$ is provable. From Lemma A and the fact that consequents can contain at most one formula, it immediately follows that $p \not\subset q \Rightarrow B$ and $p \not\subset q \Rightarrow C$ are both provable. Also $B \wedge C, A_2, \ldots, A_n \Rightarrow p \not\subset q$ and $B, C \Rightarrow B \wedge C$ are both provable, so by one application of the cut rule,

$$B, C, A_2, \ldots, A_n \Rightarrow p \not\subset q$$

is provable. But the total number of logical symbols in $B, C, A_2, \ldots, A_n$ is $<K$, contradicting the induction hypothesis. Thus we can assume that no A_i is of the form $B \wedge C$. Now suppose that each A_i is of the form $\neg A_i'$, and that the Lemma fails. Then each of the following is provable:

(*) $$\neg A_1', \ldots, \neg A_n' \Rightarrow p \not\subset q$$

(**) $$p \not\subset q \Rightarrow \neg A_i', \quad i = 1, \ldots, n.$$

Consider the last inference to (*). It cannot be $\neg$-IA. If it were a thinning on the right, repeated use of (**), the cut rule, and contraction on the left would prove $p \not\subset q \Rightarrow$, contradicting the Corollary to Lemma B. If it were a thinning on the left, the induction hypothesis would be violated.

This leaves contraction on the left or $\not\subset$ –*IS* as possibilities, and so the pattern must be:

$$\cfrac{p, \neg A'_{i_1}, \ldots, \neg A'_{i_j} \Rightarrow \qquad \neg A'_{k_1}, \ldots, \neg A'_{k_l} \Rightarrow q}{\dfrac{\neg A'_{i_1}, \ldots, \neg A'_{k_l} \Rightarrow p \not\subset q}{\neg A_1', \ldots, \neg A_n' \Rightarrow p \not\subset q}} \not\subset\text{-}IS \quad \text{possibly some contractions and interchanges.}$$

But from $\neg A'_{k_i}, \ldots, \neg A'_{k_1} \Rightarrow q$ and (**), by repeated use of the cut and contraction rules, we can prove $p \not\subset q \Rightarrow q$, contradicting Lemma C. Thus not all of the A_i can be of the form $\neg A_i'$. So some A_i, say A_1, must be either of the form $B \vee C$ or $B \supset C$. Suppose A_1 is $B \vee C$ and that again the Lemma fails. Then $p \not\subset q \Rightarrow B \vee C$ is provable, and by Lemma A, it is easy to see that either $p \not\subset q \Rightarrow B$ or $p \not\subset q \Rightarrow C$ is provable. Since $B \Rightarrow B \vee C$ and $C \Rightarrow B \vee C$ are both provable, it follows that both

$$B, A_2, \ldots, A_n \Rightarrow p \not\subset q$$

and

$$C, A_2, \ldots, A_n \Rightarrow p \not\subset q$$

are provable. Replacing A_1 by B or by C as is appropriate, we contradict the induction hypothesis. Thus each A_i is either a negation or implication. So suppose that A_1 is $B \supset C$. Then $p \not\subset q \Rightarrow B \supset C$ is provable, and so by Lemma A,

(***) $$B, p \not\subset q \Rightarrow C$$

is provable, as is

(†) $$B \supset C, A_2, \ldots, A_n \Rightarrow p \not\subset q.$$

Now we must consider the patterns of inference leading to (†). As argued above the last inference to (†) cannot be a thinning, nor can it be $\neg$ –*IA*. This, by Lemma A, leaves two possible patterns:

(††) $$\cfrac{A_{i_1}, \ldots, A_{i_j} \Rightarrow B \qquad C, A_{k_1}, \ldots, A_{k_l} \Rightarrow p \not\subset q}{\dfrac{B \supset C, A_{i_1}, \ldots, A_{k_l} \Rightarrow p \not\subset q}{B \supset C, A_2, \ldots, A_n \Rightarrow p \not\subset q}} \supset\text{-}IA \quad \text{possibly some interchanges and contractions.}$$

(‡) $$\cfrac{p, A_{i_1}, \ldots, A_{i_j} \Rightarrow \qquad A_{k_1}, \ldots, A_{k_l} \Rightarrow q}{\dfrac{A_{i_1}, \ldots, A_{k_l} \Rightarrow p \not\subset q}{A_1, \ldots, A_n \Rightarrow p \not\subset q}} \not\subset\text{-}IS \quad \text{possibly some interchanges and contractions.}$$

Now (‡) is not possible since $A_{k_1}, \ldots, A_{k_l} \Rightarrow q$ and $p \not\subset q \Rightarrow A_i$, $i = 1, \ldots, n$, lead to $p \not\subset q \Rightarrow q$, contradicting Lemma C. For (††), we first observe that from $A_{i_1}, \ldots, A_{i_j} \Rightarrow B$ and (***), by using the cut rule, we get

$$A_{i_1}, \ldots, A_{i_j}, p \not\subset q \Rightarrow C.$$

Then using $p \not\subset q \Rightarrow A_i$, $i=1, \ldots, n$, together with the cut and contraction rules, we get

$$p \not\subset q \Rightarrow C.$$

From $C, A_{k_1}, \ldots, A_{k_l} \Rightarrow p \not\subset q$, by thinning, we get

$$C, A_2, \ldots, A_n \Rightarrow p \not\subset q.$$

But now, replacing A_1 by C, we contradict the induction hypothesis. This completes the proof of the Lemma.

Theorem 2. The connective $\not\subset$ is independent of the other connectives in $LJ_{\not\subset}$.

Proof. If not, there would be a formula A not involving $\not\subset$ such that $\Rightarrow A \equiv p \not\subset q$ would be provable. By Lemma A, it would follow that both $A \Rightarrow p \not\subset q$ and $p \not\subset q \Rightarrow A$ were provable, contradicting Lemma D.

Theorem 3. If the calculus $LJ_{\not\subset}$ is interpreted in Kripke's model structures in such a way that $LJ_{\not\subset}$ is complete with respect to this interpretation and $\supset$, $\neg$, $\vee$, and $\wedge$ are given their usual interpretations, then the interpretation of $\not\subset$ cannot be a regular connective in the sense of [2].

Proof. Suppose $\not\subset$ is interpreted as a regular connective. Then there is a regular metalogical formula (cf. [2]) $\mathscr{A}(\Gamma\ p, q)$ such that for all models $<G, R, \models>$ and all $\Gamma \in G$,

$$\Gamma \models p \not\subset q \text{ iff } \mathscr{A}(\Gamma, p, q).$$

But then by Theorem 2.1 of [2], there exists a formula A involving only $\supset$, $\neg$, $\vee$, and $\wedge$ such that for all $<G, R, \models>$ and all $\Gamma \in G$,

$$\mathscr{A}(\Gamma, p, q) \text{ iff } \Gamma \models A,$$

and so

$$\Gamma \models p \not\subset q \equiv A.$$

Hence $p \not\subset q \equiv A$ is universally valid, and so if $LJ_{\not\subset}$ were complete with respect to this interpretation, $\Rightarrow p \not\subset q \equiv A$ would be provable in $LJ_{\not\subset}$, contradicting Theorem 2.

Theorem 4. If $\Rightarrow A \not\subset B$ is provable in $LJ_{\not\subset}$, then both $\Rightarrow B$ and $A \Rightarrow$ are provable in $LJ_{\not\subset}$.

Proof. This follows immediately from Lemma A.

Although $\Rightarrow \neg A \equiv A \supset (p \not\subset p)$ is provable in $LJ_{\not\subset}$, it is possible to prove analogues of Lemma D for the other binary connectives, leading to

Theorem 5. The binary connectives of $LJ_{\not\subset}$ form an independent set.

Now consider the calculus $LJ_{\not\subset, |, \downarrow}$ obtained from $LJ_{\not\subset}$ by adding the binary connectives $|$ (not both) and $\downarrow$ (neither-nor), together with the following rules for their introduction:

$|$-*IA*:
$$\frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{A|B,\, \Gamma \Rightarrow}$$

$|$-*IS*:
$$\frac{A,\, \Gamma \Rightarrow}{\Gamma \Rightarrow A|B} \qquad\qquad \frac{B,\, \Gamma \Rightarrow}{\Gamma \Rightarrow A|B}$$

$\downarrow$-*IA*
$$\frac{\Gamma \Rightarrow A}{A \downarrow B,\, \Gamma \Rightarrow} \qquad\qquad \frac{\Gamma \Rightarrow B}{A \downarrow B,\, \Gamma \Rightarrow}$$

$\downarrow$-*IS*
$$\frac{A,\, \Gamma \Rightarrow \qquad B,\, \Gamma \Rightarrow}{\Gamma \Rightarrow A \downarrow B.}$$

The proofs of the following theorems are analogous to those of Theorems 1–5, and so we omit them.

Theorem 6. The Hauptsatz holds for the system $LJ_{\not\subset,|,\downarrow}$.

Theorem 7. Although $\Rightarrow p \downarrow q \equiv \neg p \wedge \neg q$ is provable, the binary connectives of $LJ_{\not\subset,|,\downarrow}$ other than $\downarrow$ form an independent set.

Theorem 8. If the calculus $LJ_{\not\subset,|,\downarrow}$ is interpreted in Kripke's model structures in such a way that $LJ_{\not\subset,|,\downarrow}$ is complete with respect to this interpretation and $\supset$, $\neg$, $\vee$, and $\wedge$ are given their usual interpretations, then the interpretation of $|$ cannot be regular in the sense of [2]. Clearly $\downarrow$ is regular.

Theorem 9. If $\Rightarrow A|B$ is provable in $LJ_{\not\subset,|,\downarrow}$, then either $A \Rightarrow$ or $B \Rightarrow$ is provable in $LJ_{\not\subset,|,\downarrow}$. If $\Rightarrow A \downarrow B$ is provable in $LJ_{\not\subset,|,\downarrow}$, then both $A \Rightarrow$ and $B \Rightarrow$ are provable in $LJ_{\not\subset,|,\downarrow}$.

Theorems 4 and 9 indicate that it may be reasonable to call the connectives $\not\subset$, $|$, $\downarrow$ introduced here intuitionistic connectives. For if we are to have evidence that A is not implied by B, what better evidence can we ask than evidence for B and evidence that A is absurd? Again, Theorem 9 indicates that evidence for $A|B$ consists either in evidence that A is absurd or in evidence that B is absurd, and that evidence for $A \downarrow B$ consists in evidence that A is absurd and evidence that B is absurd.

Since the sequentzen systems are constructed so as to separate the roles of each logical particle, it appears evident that these new connectives may be added to the full system LJ while still maintaining the Hauptsatz. Finally, it also appears that imitation of the foregoing constructions should allow one to construct similar n-place connectives for arbitrary n (and perhaps even different sorts of quantifiers) all of which would be reasonable candidates for intuitionistic connectives and which would all be independent of one another.

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